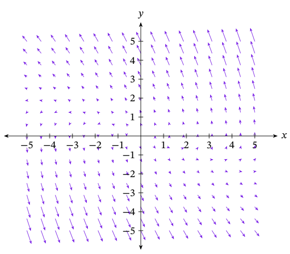
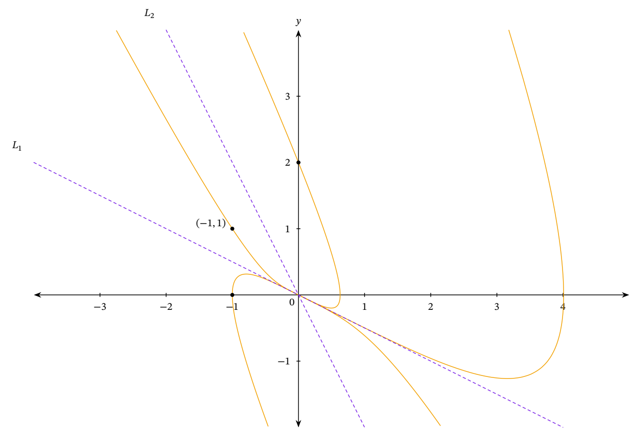
Worksheet: Integral Curves of Vector Fields

Q1: The figures show the vector field $\left\langle -y, x + \frac{5}{2}y \right\rangle$, together with several of its flows.





Suppose we know that for some numbers k the integral curves $x = f(t), y = g(t)$ are such that f and g are linear combinations of some e^{kt}. What are the values of k?

A $\frac{1}{2}$ and 2

B $\frac{1}{3}$ and 2

C $\frac{1}{3}$ and 3

D $\frac{1}{2}$ and 3

E $\frac{1}{4}$ and 2

▸ What are the parametric equations of the integral curve that is at $(-1,0)$ when $t=0$?

A $x=-\frac{4}{3}e^{\frac{t}{2}}+\frac{1}{3}e^{2t}, y=\frac{2}{3}e^{\frac{t}{2}}-\frac{2}{3}e^{2t}$

B $x=\frac{4}{3}e^{\frac{t}{3}}+\frac{1}{3}e^{2t}, y=\frac{2}{3}e^{\frac{t}{2}}-\frac{2}{3}e^{2t}$

C $x=\frac{4}{3}e^{\frac{t}{2}}-\frac{1}{3}e^{2t}, y=-\frac{2}{3}e^{\frac{t}{2}}+\frac{2}{3}e^{2t}$

D $x=-\frac{4}{3}e^{\frac{t}{2}}+\frac{1}{3}e^{2t}, y=-\frac{2}{3}e^{\frac{t}{2}}+\frac{2}{3}e^{2t}$

E $x=-\frac{4}{3}e^{\frac{t}{3}}+\frac{1}{3}e^{2t}, y=-\frac{2}{3}e^{\frac{t}{2}}+\frac{2}{3}e^{2t}$

▸ What are the parametric equations of the integral curve that is at $(0,2)$ when $t=0$?

A $x=\frac{4}{3}e^{\frac{t}{2}}-\frac{4}{3}e^{2t}, y=-\frac{2}{3}e^{\frac{t}{2}}+\frac{8}{3}e^{2t}$

B $x=\frac{4}{3}e^{\frac{t}{3}}+\frac{4}{3}e^{2t}, y=-\frac{2}{3}e^{\frac{t}{2}}+\frac{8}{3}e^{2t}$

C $x=-\frac{4}{3}e^{\frac{t}{2}}-\frac{4}{3}e^{2t}, y=\frac{2}{3}e^{\frac{t}{2}}-\frac{8}{3}e^{2t}$

D $x=\frac{4}{3}e^{\frac{t}{2}}+\frac{4}{3}e^{2t}, y=\frac{2}{3}e^{\frac{t}{2}}+\frac{8}{3}e^{2t}$

E $x=\frac{4}{3}e^{\frac{t}{3}}-\frac{4}{3}e^{2t}, y=\frac{2}{3}e^{\frac{t}{2}}-\frac{8}{3}e^{2t}$

▸ What are the parametric equations of the integral curve that is at $(-1, 1)$ when $t = 0$?

A $x = -\frac{2}{3}e^{\frac{t}{2}} - \frac{e^{2t}}{3}, y = \frac{e^{\frac{t}{2}}}{3} + \frac{2}{3}e^{2t}$

B $x = -\frac{2}{3}e^{\frac{t}{3}} - \frac{e^{2t}}{3}, y = \frac{e^{\frac{t}{2}}}{3} - \frac{2}{3}e^{2t}$

C $x = \frac{2}{3}e^{\frac{t}{2}} + \frac{e^{2t}}{3}, y = -\frac{e^{\frac{t}{2}}}{3} + \frac{2}{3}e^{2t}$

D $x = -\frac{2}{3}e^{\frac{t}{2}} + \frac{e^{2t}}{3}, y = \frac{e^{\frac{t}{2}}}{3} - \frac{2}{3}e^{2t}$

E $x = -\frac{2}{3}e^{\frac{t}{3}} + \frac{e^{2t}}{3}, y = -\frac{e^{\frac{t}{2}}}{3} + \frac{2}{3}e^{2t}$

▸ As $t \to \infty$ and as $t \to -\infty$ along an integral curve, the secant between $(0, 0)$ and $(f(t), g(t))$ approaches one of the lines L_1 and L_2 shown. What are the slopes of these two lines?

A slope of $L_1 = -\frac{1}{2}$, slope of $L_2 = -2$

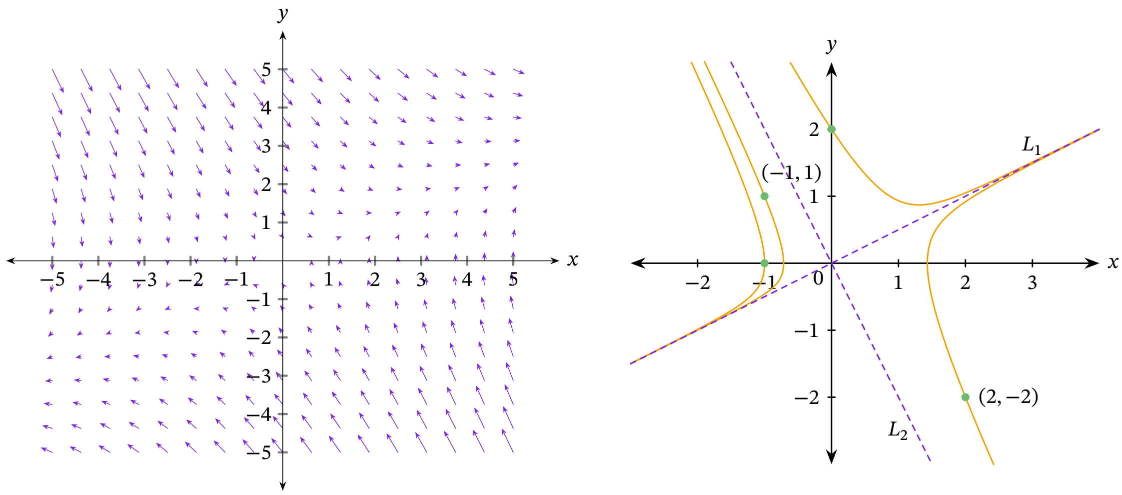
B slope of $L_1 = -\frac{1}{4}$, slope of $L_2 = -2$

C slope of $L_1 = \frac{1}{2}$, slope of $L_2 = 2$

D slope of $L_1 = -\frac{1}{2}$, slope of $L_2 = 2$

E slope of $L_1 = \frac{1}{4}$, slope of $L_2 = 2$

Q2: The figures show the vector field $\left\langle y, x - \frac{3}{2}y \right\rangle$, together with several of its flows.



▸ Suppose we know that, for some numbers k, the integral curves $x = f(t)$ and $y = g(t)$ are such that f and g are linear combinations of some e^{kt}. What are the values of k?

A $\frac{1}{2}$ and $\frac{1}{4}$

B $\frac{1}{4}$ and -2

C $\frac{1}{2}$ and -2

D $\frac{1}{4}$ and 2

E $\frac{1}{2}$ and 2

▸ What are the parametric equations of the integral curve that is at $(0, 2)$ when $t = 0$?

A $x = \frac{3}{5}e^{\frac{t}{2}} - \frac{4}{5}e^{-2t}, y = \frac{3}{5}e^{\frac{t}{2}} + \frac{8}{5}e^{-2t}$

B $x = \frac{4}{5}e^{\frac{t}{2}} - \frac{4}{5}e^{2t}, y = \frac{2}{5}e^{\frac{t}{2}} + \frac{8}{5}e^{2t}$

C $x = \frac{4}{5}e^{\frac{t}{2}} - \frac{4}{5}e^{-2t}, y = \frac{2}{5}e^{\frac{t}{2}} + \frac{8}{5}e^{-2t}$

D $x = \frac{4}{5}e^{\frac{t}{2}} + \frac{4}{5}e^{2t}, y = \frac{2}{5}e^{\frac{t}{2}} - \frac{8}{5}e^{2t}$

E $x = \frac{4}{5}e^{\frac{t}{2}} + \frac{4}{5}e^{-2t}, y = \frac{2}{5}e^{\frac{t}{2}} - \frac{8}{5}e^{-2t}$

▸ What are the parametric equations of the integral curve that is at $(-1, 1)$ when $t = 0$?

A $x = -\frac{1}{5}e^{\frac{t}{2}} - \frac{3}{5}e^{-2t}, y = -\frac{2}{5}e^{\frac{t}{2}} - \frac{6}{5}e^{-2t}$

B $x = -\frac{2}{5}e^{\frac{t}{2}} - \frac{3}{5}e^{2t}, y = -\frac{1}{5}e^{\frac{t}{2}} + \frac{6}{5}e^{2t}$

C $x = -\frac{2}{5}e^{\frac{t}{2}} - \frac{3}{5}e^{-2t}, y = -\frac{1}{5}e^{\frac{t}{2}} + \frac{6}{5}e^{-2t}$

D $x = -\frac{2}{5}e^{\frac{t}{2}} + \frac{3}{5}e^{2t}, y = -\frac{1}{5}e^{\frac{t}{2}} - \frac{6}{5}e^{2t}$

E $x = -\frac{2}{5}e^{\frac{t}{2}} + \frac{3}{5}e^{-2t}, y = -\frac{1}{5}e^{\frac{t}{2}} - \frac{6}{5}e^{-2t}$

▸ What are the parametric equations of the integral curve that is at $(2, -2)$ when $t = 0$?

A $x = \frac{3}{5}e^{\frac{t}{2}} - \frac{6}{5}e^{-2t},\ y = \frac{3}{5}e^{\frac{t}{2}} - \frac{12}{5}e^{-2t}$

B $x = \frac{4}{5}e^{\frac{t}{2}} + \frac{6}{5}e^{2t},\ y = \frac{2}{5}e^{\frac{t}{2}} - \frac{12}{5}e^{2t}$

C $x = \frac{4}{5}e^{\frac{t}{2}} + \frac{6}{5}e^{-2t},\ y = \frac{2}{5}e^{\frac{t}{2}} - \frac{12}{5}e^{-2t}$

D $x = \frac{4}{5}e^{\frac{t}{2}} - \frac{6}{5}e^{2t},\ y = \frac{2}{5}e^{\frac{t}{2}} + \frac{12}{5}e^{2t}$

E $x = \frac{4}{5}e^{\frac{t}{2}} - \frac{6}{5}e^{-2t},\ y = \frac{2}{5}e^{\frac{t}{2}} + \frac{12}{5}e^{-2t}$

▸ Using the fact that $\left(e^{\frac{t}{2}}\right)^4 \cdot e^{-2t} = 1$, find a Cartesian equation satisfied by the points of the integral curve that is at $(0, 2)$ when $t = 0$. You need not simplify your expression.

A $\left(x + \frac{1}{3}y\right)^4 \left(\frac{1}{2}x - \frac{1}{6}y\right) = 1$

B $\left(x + \frac{1}{6}y\right)^4 \left(\frac{1}{2}x - \frac{1}{3}y\right) = 1$

C $\left(x + \frac{1}{2}y\right)^4 \left(\frac{1}{6}x - \frac{1}{3}y\right) = 1$

D $\left(x + \frac{1}{6}y\right)^4 \left(\frac{1}{2}x + \frac{1}{3}y\right) = 1$

E $\left(x + \frac{1}{2}y\right)^4 \left(\frac{1}{6}x + \frac{1}{3}y\right) = 1$

As $t \to \infty$ and $t \to -\infty$ along an integral curve, the secant between $(0, 0)$ and $(f(t), g(t))$ approaches one of the lines L_1 and L_2 shown. What are the slopes of these two lines?

A slope of $L_1 = \frac{1}{2}$, slope of $L_2 = 1$

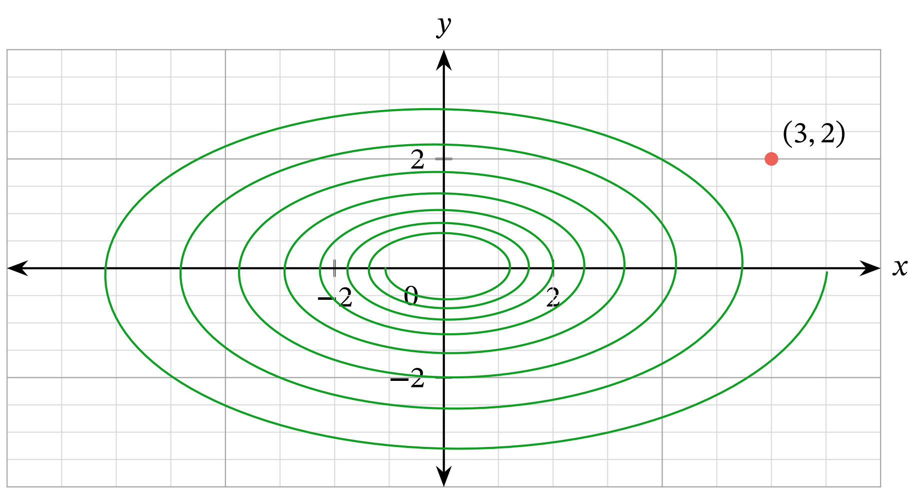
B slope of $L_1 = \frac{1}{3}$, slope of $L_2 = -2$

C slope of $L_1 = \frac{1}{2}$, slope of $L_2 = -2$

D slope of $L_1 = \frac{1}{4}$, slope of $L_2 = -1$

E slope of $L_1 = \frac{1}{2}$, slope of $L_2 = 2$

Q3: Consider the parametric curve $x = e^{at}\cos(bt)$, $y = e^{at}\sin(bt)$ with constants a and b. The figure shows the case $a = \frac{1}{5}$ and $b = 5$ for $-\pi \le t \le 2\pi$.



▸ Find a vector field such that the curve $x = e^{at}\cos(bt)$ and $y = e^{at}\sin(bt)$ is its integral curve.

A $\langle ax + by, bx - ay \rangle$

B $\langle bx + ay, ax - by \rangle$

C $\langle ax - by, ax + by \rangle$

D $\langle bx - ay, bx + ay \rangle$

E $\langle ax - by, bx + ay \rangle$

▸ Find a linear second-order differential equation satisfied by x.

A $2x'' + ax' + (a^2 - b^2)\,x = 0$

B $x'' - ax' + (2a^2 + b^2)\,x = 0$

C $x'' - ax' + (a^2 + 2b^2)\,x = 0$

D $x'' + ax' + (a^2 - b^2)\,x = 0$

E $x'' - 2ax' + (a^2 + b^2)\,x = 0$

You can check that $x = e^{at}\sin(bt)$ is also a solution to this differential equation and therefore any $x = f(t) = Pe^{at}\cos(bt) + Qe^{at}\sin(bt)$ for constants P and Q. Using the vector field, determine the corresponding function $y = g(t)$ so that $x = f(t)$ and $y = g(t)$ is an integral curve.

A $Qe^{at}\cos(at) + Pe^{at}\sin(at)$

B $-Qe^{bt}\cos(at) + Pe^{bt}\sin(at)$

C $Qe^{at}\cos(bt) + Pe^{at}\sin(bt)$

D $-Qe^{bt}\cos(bt) + Pe^{at}\sin(at)$

E $-Qe^{at}\cos(bt) + Pe^{at}\sin(bt)$

For the case $a = \frac{1}{5}$ and $b = 5$, find parametric equations for the integral curve that starts at the point $(3, 2)$ when $t = 0$.

A $x = 3e^{\frac{t}{5}}\cos(5t) + 2e^{\frac{t}{5}}\sin(5t),\ y = 2e^{\frac{t}{5}}\cos(5t) - 3e^{\frac{t}{5}}\sin(5t)$

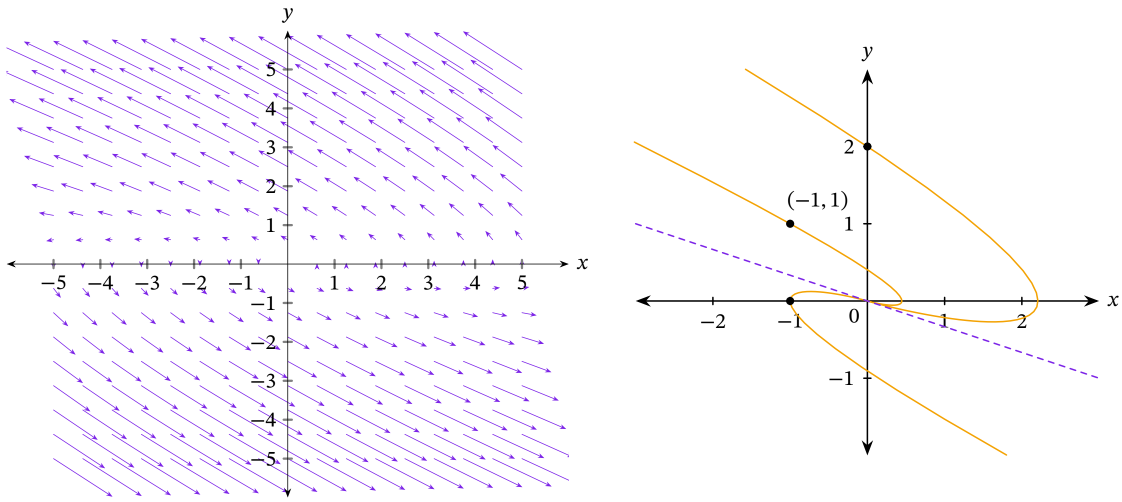
B $x = 2e^{2\frac{t}{5}}\cos(5t) + 2e^{\frac{t}{5}}\sin(5t),\ y = 3e^{\frac{t}{5}}\cos(5t) + 3e^{\frac{t}{5}}\sin(5t)$

C $x = 2e^{2\frac{t}{5}}\cos(5t) - 2e^{\frac{t}{5}}\sin(5t),\ y = 3e^{2\frac{t}{5}}\cos(5t) + 3e^{\frac{t}{5}}\sin(5t)$

D $x = e^{\frac{t}{5}}\cos(5t) - 2e^{\frac{t}{5}}\sin(5t),\ y = 3e^{\frac{t}{5}}\cos(5t) + 3e^{\frac{t}{5}}\sin(5t)$

E $x = 3e^{\frac{t}{5}}\cos(5t) - 2e^{\frac{t}{5}}\sin(5t),\ y = 2e^{\frac{t}{5}}\cos(5t) + 3e^{\frac{t}{5}}\sin(5t)$

Q4: The figures show the vector field $\langle -9y, x + 6y \rangle$, together with several of its flows.



▸ Suppose we know that, for some numbers k, the integral curves $x = f(t)$ and $y = g(t)$ are such that f and g are linear combinations of some e^{kt}. What are the values of k?

- A 2
- B 3
- C 4
- D 1
- E 6

In this case, where k is a repeated root, linear combinations of te^{kt} and e^{kt} are used. Hence, find the parametric equations of the integral curve that is at $(0, 2)$ when $t = 0$.

- A $x = 18te^{3t}$, $y = 2e^{3t} - 6te^{3t}$
- B $x = -18te^{3t}$, $y = 2e^{3t} + 6te^{3t}$
- C $x = -9e^{3t}$, $y = 2e^{3t} - 3te^{3t}$
- D $x = -9te^{3t}$, $y = 2e^{3t} + 3te^{3t}$
- E $x = -3te^{3t}$, $y = 2e^{3t} + 2te^{3t}$

What are the parametric equations of the integral curve that is at $(-1, 1)$ when $t = 0$?

- A $x = e^{3t} - 6te^{3t}$, $y = e^{3t} - 2te^{3t}$
- B $x = -e^{3t} - 6te^{3t}$, $y = e^{3t} + 2te^{3t}$
- C $x = -e^{2t} + 6te^{2t}$, $y = e^{2t} - 2te^{2t}$
- D $x = -e^{3t} - 2te^{3t}$, $y = e^{3t} + te^{3t}$
- E $x = -e^{2t} - 6te^{2t}$, $y = e^{2t} + 2te^{2t}$

▸ What are the parametric equations of the integral curve that is at $(-1, 0)$ when $t = 0$?

A $x = -e^{3t} - 3te^{3t}, y = te^{3t}$

B $x = -e^{3t} + 3te^{3t}, y = -te^{3t}$

C $x = -e^{3t} - e^{3t}, y = te^{3t}$

D $x = -e^{3t} - 2te^{3t}, y = -te^{3t}$

E $x = -e^{3t} - 3te^{3t}, y = te^{3t}$

▸ As $t \to \infty$ and $t \to -\infty$ along an integral curve, the secant between (0, 0) and $(f(t), g(t))$ approaches the dashed line shown. What is the slope of this line?

A $\frac{1}{3}$

B $-\frac{1}{3}$

C $\frac{1}{2}$

D $-\frac{1}{4}$

E $-\frac{1}{2}$

Q5: If $x = f(t)$ parameterizes an integral curve of the vector field $V(x, y) = \langle y, x \rangle$, then $f'' = f$. This means f is a linear combination of e^t and e^{-t}.

▸ Find the x-parameter function $f(t)$ for the integral curve to this vector field that starts at the point $(2, 3)$.

A $f(t) = \frac{1}{2}e^t + \frac{5}{2}e^{-t}$

B $f(t) = \frac{5}{2}e^t + \frac{1}{2}e^{-t}$

C $f(t) = \frac{3}{2}e^t - \frac{2}{3}e^{-t}$

D $f(t) = \frac{1}{2}e^t - \frac{5}{2}e^{-t}$

E $f(t) = \frac{5}{2}e^t - \frac{1}{2}e^{-t}$

▸ Find the Cartesian equation of the integral curve determined above.
Hint : It is a hyperbola.

A $x^2 + y^2 = 5$

B $x^2 - y^2 = 5$

C $x^2 - 2y^2 = -5$

D $x^2 + y^2 = -5$

E $x^2 - y^2 = -5$

▸ Find the Cartesian equation of the integral curve to this vector field that starts at the point $(2, 2)$.

A $y = x^2$

B $y = x + 2$

C $y = -x^2$

D $y = -x$

E $y = x$

Q6: An integral curve (or flow) of a vector field V is a parametric curve $x = f(t), y = g(t)$ with $\langle f'(t), g'(t) \rangle = V(f(t), g(t))$ for every t where f and g are defined.

▸ By solving the equations $f'(t) = 1$ and $g'(t) = 2$, find an integral curve for the vector field $V(x, y) = \langle 1, 2 \rangle$ that also satisfies $(f(0), g(0)) = (-1, 1)$.

A $f(t) = 2t - 1, g(t) = 2t + 1$

B $f(t) = 2t + 1, g(t) = t - 1$

C $f(t) = t - 1, g(t) = 2t - 1$

D $f(t) = t + 1, g(t) = 2t + 1$

E $f(t) = t - 1, g(t) = 2t + 1$

Consider the vector field $V(x, y) = \langle 1, 2 \rangle$. Find the Cartesian equation of the vector field's integral curve which is at the point $(2, -3)$ when $t = 0$.

A $y - x = -7$

B $2y - x = -7$

C $y + 2x = -7$

D $y - 2x = 7$

E $y - 2x = -7$

Find the Cartesian equation of the integral curve to $V(x, y) = \langle 1, x \rangle$ that starts at the point (2, 2).

A $y = \frac{x^2}{3}$

B $y = x^2 - 2$

C $y = \frac{x^2 - 5x + 10}{2}$

D $y = \frac{x^2 - 4x + 8}{2}$

E $y = \frac{x^2}{2}$

▸ Find the Cartesian equation of the integral curve to $V(x, y) = \langle x, x^2 \rangle$ that starts at the point (2, 2).

A $y = \dfrac{x^2}{3}$

B $y = x^2 - 2$

C $y = \dfrac{x^2 - 4x + 8}{2}$

D $y = \dfrac{(x-1)^2}{2} + 2(x-1) + \ln(x-1) - \dfrac{1}{2}$

E $y = \dfrac{x^2}{2}$

▸ Find the parametric equations of the integral curve to $V(x, y) = \langle x, x^2 \rangle$ that starts at the point (0, 2).

A $f(t) = -1, g(t) = 3$

B $f(t) = 1, g(t) = 0$

C $f(t) = -1, g(t) = 2$

D $f(t) = 0, g(t) = 3$

E $f(t) = 0, g(t) = 2$

▸ Do the integral curves of the vector fields $\langle 1, x \rangle$ and $\langle x, x^2 \rangle$ starting at (0, 2) describe the same set in $\mathbb{R}^2$ for $t \geq 0$?

A no

B yes

▸ Do the integral curves of the vector fields $\langle 1, x\rangle$ and $\langle x, x^2\rangle$ starting at (2,2) describe the same set in $\mathbb{R}^2$ for $t \geq 0$?

A yes

B no

▸ The integral curves to $\langle 1, x\rangle$ and $\langle x, x^2\rangle$ that start at $(-2, 4)$ both lie inside the curve $y = \dfrac{x^2}{2} + 2$ but go in opposite directions. Determine the parametric equations that integrate the vector field $\langle x^2, x^3\rangle$ and start at $(-2, 4)$.

A $f(t) = \dfrac{-2}{t+1}, g(t) = \dfrac{2}{(t+1)^2} + 5$

B $f(t) = \dfrac{-2}{2t+1}, g(t) = \dfrac{3}{(2t+3)^2} + 5$

C $f(t) = \dfrac{-2}{2t+1}, g(t) = \dfrac{2}{(2t+1)^2} + 2$

D $f(t) = \dfrac{-3}{2t+3}, g(t) = \dfrac{2}{(2t+1)^2} + 2$

▸ Do the integral curves of the vector fields $\langle 1, x\rangle$ and $\langle x^2, x^3\rangle$ starting at (0, 2) describe the same set in $\mathbb{R}^2$ for $t \geq 0$?

A yes

B no