Worksheet: Formal Definition of a Limit

Q1: Find the largest $\delta > 0$ such that if $|x - 5| < \delta$, then $\left|\frac{1}{x} - \frac{1}{5}\right| < \frac{1}{10}$. Give your answer as a fraction.

A $\frac{53}{3}$

B $\frac{5}{3}$

C $\frac{47}{10}$

D $\frac{25}{3}$

E $\frac{1}{255}$

Q2: Find the largest $\delta > 0$ such that if $|x - 5| < \delta$, then $\left|\frac{1}{x} - \frac{1}{5}\right| < \varepsilon$. Give your answer as a fraction involving ε.

A $\frac{5}{5\varepsilon + 1}$

B $-\frac{25\varepsilon + 10}{5\varepsilon + 1}$

C $-\frac{25\varepsilon}{5\varepsilon + 1}$

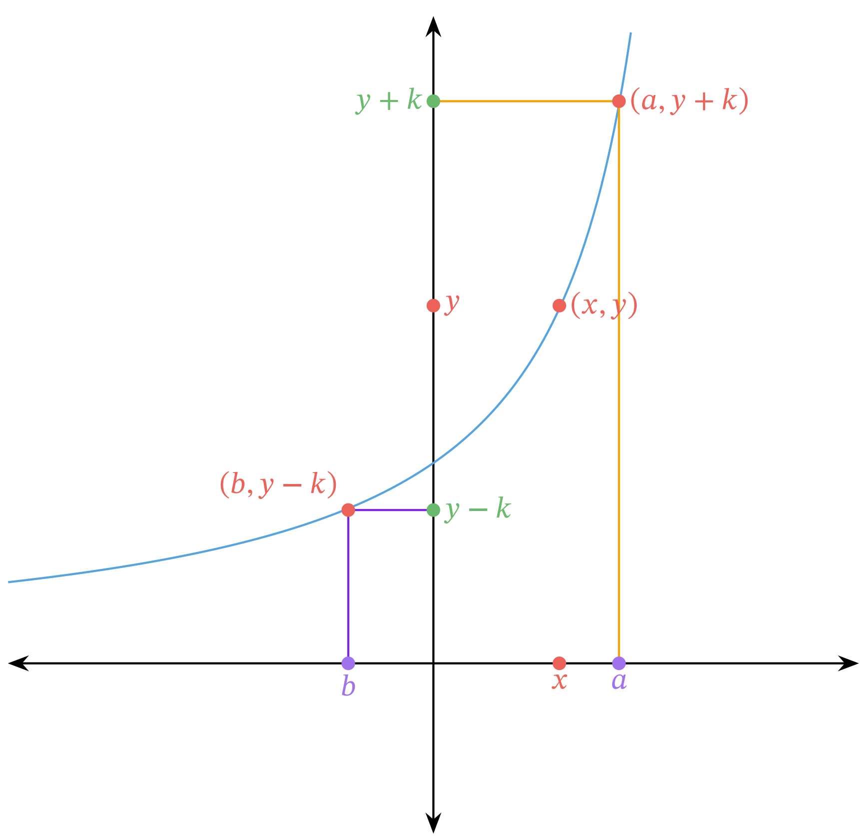
D $\frac{25\varepsilon + 10}{5\varepsilon + 1}$

E $\frac{25\varepsilon}{5\varepsilon + 1}$

Q3: Find the largest $a > 0$ such that if $|x - a| < a$, then $\left|\frac{1}{x} - \frac{1}{a}\right| < \varepsilon$. Give your answer as a fraction involving ε and a.

- A $\dfrac{a\varepsilon - 1}{a^2\varepsilon}$
- B $\dfrac{a\varepsilon + 1}{a^2\varepsilon}$
- C $\dfrac{a^2\varepsilon - 2a}{a\varepsilon + 1}$
- D $\dfrac{a^2\varepsilon + 2a}{a\varepsilon + 1}$
- E $\dfrac{a^2\varepsilon}{a\varepsilon + 1}$

Q4: In the figure, we have the graph $y = f(x)$ that is increasing and concave up. Next to (x, y) are the points $(a, y + k)$ and $(b, y - k)$ for a small $k > 0$.



▸ Which point is closer to x along the horizontal axis, a or b?

A a

B b

▸ What is the distance δ between the nearest point and x from your answer above? Give an expression that involves f and absolute values.

A $|x - f^{-1}(f(x) + k)|$

B $|x + f^{-1}(f(x) - k)|$

C $|x + f^{-1}(f(x) + k)|$

D $|x - f^{-1}(f(x) - k)|$

▸ Use your answers above to find δ such that $|e^x - 1| < 0.1$ whenever $|x - 0| < \delta$. Give your answer to 4 decimal places.

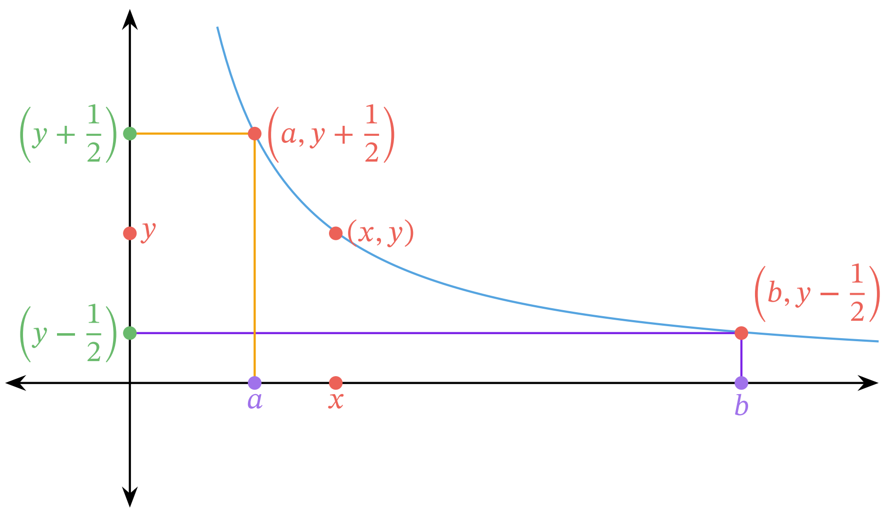
A 0.0953

B 0.1054

C −0.0953

D −0.1054

Q5: The figure shows the graph of $f(x) = \dfrac{2}{x}$ around a point (x, y) where $x > 0$ and $y = f(x)$. Nearby are points $\left(a, y + \dfrac{1}{2}\right)$ and $\left(b, y - \dfrac{1}{2}\right)$.



▸ From the graph, looking at the x-axis, which of the points a and b is nearest to x? Let this number be p.

A b

B a

▸ What is a in terms of x?

A $\dfrac{4+x}{4x}$

B $\dfrac{4x}{4+x}$

C $\dfrac{4+x}{2x}$

D $\dfrac{2x}{4+x}$

E $\dfrac{4}{x}$

▸ The claim is that if $x > 0$ and $\frac{2}{x} > \frac{1}{2}$, then $\left|\frac{2}{x^*} - \frac{2}{x}\right| < \frac{1}{2}$ provided that $|x^* - x| < \delta$ is true for all small δ. What is the largest such δ?

A $\frac{x+4}{x^2}$

B $\frac{x^2}{x+4}$

C $\frac{x}{x+4}$

D $\frac{x+4}{x}$

E $\frac{x^2}{x+2}$

▸ The condition $\frac{2}{x} > \frac{1}{2}$ was to ensure truth to the figure. Does the same δ work if $x \geq 4$?

A no

B yes

▸ It would appear that all your arguments depended upon the fact that the graph of the function f was concave up. Let $f(x) = e^{-x}$. Find $\delta > 0$ so that $|f(x^*) - f(x)| < \frac{1}{4}$ whenever $|x^* - x| < \delta$. Assume that x is positive.

A $\delta = \ln(4 + e^{-x}) - \ln(4)$

B $\delta = \ln(4 + e^{x}) - \ln(4)$

C $\delta = 2x - \ln 4 + \ln(4 + e^{x})$

D $\delta = 2x - \ln 4 + \ln(4 + e^{-x})$

E $\delta = \ln(4 + e^{x}) + \ln(4)$

Q6: Sometimes, we do not have concavity to help determine an explicit δ that shows continuity at a point. The figure shows the graph of

$$f(x) = \begin{cases} 3 + x + x^2 & x > 0, \\ 3 + x - 4x^2 & x \leq 0, \end{cases}$$

together with its tangent line $y = x + 3$ at the inflection point $(0, 3)$.

▸ What is $f^{-1}(x)$ for $x > 0$?

A $-\frac{\sqrt{4x - 11} + 1}{2}$

B $\frac{\sqrt{4x - 11} - 1}{2}$

C $\frac{1 - \sqrt{49 - 16x}}{8}$

D $\frac{1 + \sqrt{49 - 16x}}{8}$

▸ What is $f^{-1}(x)$ for $x \leq 0$?

A $\dfrac{1+\sqrt{49-16x}}{8}$

B $\dfrac{1-\sqrt{49-16x}}{8}$

C $\dfrac{\sqrt{4x-11}-1}{2}$

D $-\dfrac{\sqrt{4x-11}+1}{2}$

▸ By considering the graph near $(0, 3)$, find the largest δ so that $|f(x)-3| < \varepsilon$ whenever $|x| < \delta$.

A $\dfrac{1-\sqrt{16\varepsilon+1}}{8}$

B $\dfrac{\sqrt{16\varepsilon+1}-1}{8}$

C $\dfrac{\sqrt{4\varepsilon+1}-1}{2}$

D $\dfrac{1-\sqrt{4\varepsilon}+1}{2}$

Q7: The graph of $f(x) = 5-(x-2)^2$ is concave down and decreasing when $x > 2$. We want to find the maximal δ so that, for a given $\varepsilon > 0$, it follows that if $|x-3| < \delta$ then $|f(x)-f(3)| < \varepsilon$. This δ will, of course, be a function of ε.

▸ What is the inverse function near $x = 3$? Give an expression for $f^{-1}(x)$.

A $2 - \sqrt{5 + x}$

B $2 + \sqrt{x - 5}$

C $2 + \sqrt{5 - x}$

D $2 + \sqrt{5 + x}$

E $2 - \sqrt{5 - x}$

▸ Using the concavity of the graph of f, determine which point is closer to 3, $f^{-1}(4 + \varepsilon)$ or $f^{-1}(4 - \varepsilon)$, along the horizontal axis. (Do not evaluate them and consider only small ε.)

A $f^{-1}(4 + \varepsilon)$

B $f^{-1}(4 - \varepsilon)$

▸ From your answers above, find an expression — in terms of ε — for the largest δ so that if $|x - 3| < \delta$ then $4 - \varepsilon < f(x) < 4 + \varepsilon$.

A $2 + \sqrt{1 + \varepsilon}$

B $\sqrt{1 + \varepsilon} - 1$

C $2 + \sqrt{1 - \varepsilon}$

D $1 - \sqrt{1 - \varepsilon}$