Explainer: Equation of a Circle Passing through Three Points

In this explainer, we will learn how to find the equation of a circle passing through three given points.

You should know the two forms of the general equation of a circle.

Equation of a Circle in Radius-Center Form

The equation of a circle centered at $C(h, k)$ and of radius r is

$$(x-h)^2+(y-k)^2=r^2.$$

Equation of a Circle in General Form

The equation of a circle can be written in the form

$$x^2+y^2+ax+by+c=0.$$

How to: Write the Equation of a Circle That Passes through Three Points in the Form $(- h)^2 + (-)^2 = {}^2$

1. The coordinates of the three given points on the circle must satisfy the equation of the circle. With $P_1(x_1, y_1)$, $P_2(x_2, y_2)$, and $P_3(x_3, y_3)$ the three given points, we can write

$$(x_1-h)^2+(y_1-k)^2=r^2,$$
$$(x_2-h)^2+(y_2-k)^2=r^2,$$
$$(x_3-h)^2+(y_3-k)^2=r^2.$$

   Note that given $c = r^2$, the above equations are simply saying that the distance between the center of the circle, of coordinates (h, k), and each of the three points is constant and equal to the radius.

2. Since the three equations are in the form Expression $= c$, they can be rearranged in a system of two equations with two unknowns by writing $E_1 = E_2$ and $E_1 = E_3$. (Note that any two combinations of the three equations work.) We get

$$(x_1-h)^2+(y_1-k)^2=(x_2-h)^2+(y_2-k)^2,$$
$$(x_1-h)^2+(y_1-k)^2=(x_3-h)^2+(y_3-k)^2.$$

   By expanding the brackets, we see that the terms h^2 and k^2 cancel out:

$$x_1^2-2x_1h+y_1^2-2y_1k=x_2^2-2x_2h+y_2-2y_2k,$$

$$x_1^2 - 2x_1h + y_1^2 - 2y_1k = x_3^2 - 2x_3h + y_3 - 2y_3k.$$

By solving this system of equations, we find the coordinates of the circle center (h, k).

3. The last stage is to plug in these values of h and k in one of our first three equations to find the value of r^2.
4. The equation of the circle is then $(x - h)^2 + (y - k)^2 = r^2$ with the values of h, k, and c we have found.

Let's see how this method is implemented when we have the coordinates of the three points.

Example 1: Writing the Equation of a Circle That Passes through Three Points in Center-Radius Form

Find the equation of the circle that passes through the points $A(8, 7)$, $B(1, 8)$, and $C(0, 1)$.

A. $(x - 8)^2 + (y - 8)^2 = 10$

B. $(x - 4)^2 + (y - 4)^2 = 25$

C. $(x + 4)^2 + (y + 4)^2 = 25$

D. $x^2 + (y - 1)^2 = 50$

Answer

1. We are looking for the equation of the circle in the form $(x - h)^2 + (y - k)^2 = r^2$. Let's write that the coordinates of the three points satisfy the equation of the circle:

$$(8 - h)^2 + (7 - k)^2 = r^2,$$
$$(1 - h)^2 + (8 - k)^2 = r^2,$$
$$(0 - h)^2 + (1 - k)^2 = r^2.$$

2. We arrange this system of three equations in a system of two equations:

$$(8 - h)^2 + (7 - k)^2 = (1 - h)^2 + (8 - k)^2,$$
$$(8 - h)^2 + (7 - k)^2 = (0 - h)^2 + (1 - k)^2.$$

By expanding the brackets, we find

$$64 - 16h + h^2 + 49 - 14k + k^2 = 1 - 2h + 64 - 16k + k^2,$$
$$64 - 16h + h^2 + 49 - 14k + k^2 = h^2 + 1 - 2k + k^2.$$

By rearranging and collecting like terms, we get

$$-14h + 2k = -48,$$
$$-2h - 14k = -64.$$

From the second equation, we get $h = 32 + 7k$. By plugging in this value into the first equation, we find $k = 4$. And by plugging this value of k in one of the two equations above, we find $h = 4$.

3. By plugging in these values of h and k in one of the first three equations, for instance, $(0-h)^2 + (1-k)^2 = r^2$, we have

$$(0+4)^2 + (1-4)^2 = r^2,$$

and thus

$$r^2 = 25.$$

4. The equation of the circle is: $(x-4)^2 + (y-4)^2 = 25$.

Let's see now another method to find the equation of a circle that passes through three points when we want to have the equation in the form $x^2 + y^2 + ax + by + c = 0$.

How to: Write the Equation of a Circle That Passes through Three Points in the Form $^2 + ^2 + + + = 0$

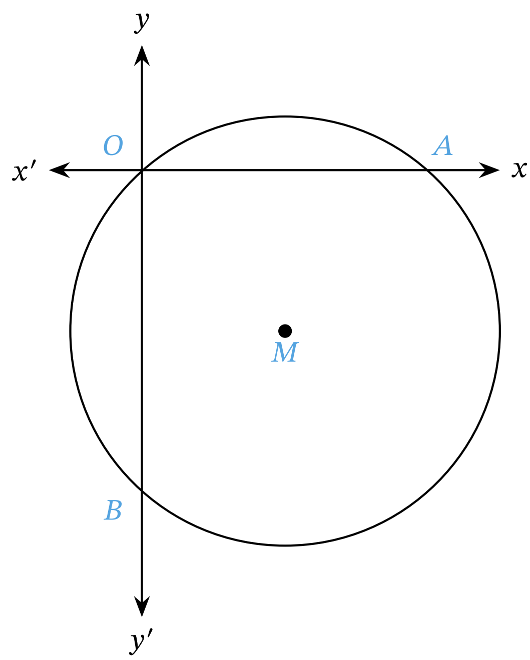
1. The coordinates of the three given points on the circle must satisfy the equation of the circle. With $P_1(x_1, y_1)$, $P_2(x_2, y_2)$, and $P_3(x_3, y_3)$ the three given points, we can write

$$x_1^2 + y_1^2 + ax_1 + by_1 + c = 0,$$
$$x_2^2 + y_2^2 + ax_2 + by_2 + c = 0,$$
$$x_3^2 + y_3^2 + ax_3 + by_3 + c = 0.$$

2. Solve the system of three equations to find the unknowns a, b, and c.
3. Plug the values found in the general equation $x^2 + y^2 + ax + by + c = 0$.

Example 2: Writing the Equation of a Circle That Passes through Three Points in General Form

Determine the general equation of the shown circle M passing through the origin point and the two points $A(8, 0)$ and $B(0, -10)$.



A. $x^2 + y^2 - 16x + 20y = 0$

B. $x^2 + y^2 - 8x + 10y = 0$

C. $x^2 + y^2 + 10x - 8y = 0$

D. $x^2 + y^2 + 8x - 10y = 0$

Answer

1. Here, we are asked to find the equation of the circle in the form $x^2 + y^2 + ax + by + c = 0$. The coordinates of the three given points on the circle must satisfy the equation of the circle. We, therefore, plug the three pairs of coordinates in the general equation to get a system of three equations:

$$8^2 + 0^2 + 8a + 0b + c = 0,$$
$$0^2 + (-10)^2 + 0a - 10b + c = 0,$$
$$0^2 + 0^2 + 0a + 0b + c = 0.$$

2. From the last equation, we find that $c = 0$. This leaves us with

$$64 + 8a = 0,$$
$$100 - 10b = 0.$$

We find

$$a = -8,$$
$$b = 10.$$

3. The equation of the circle is $x^2 + y^2 - 8x + 10y = 0$.